

A Survey of One-Dimensional Random Polymers

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In the last decade there has been an enormous progress in the mathematical understanding of one-dimensional polymer measures, which are path measures that suppress self-intersections. We are currently in the situation that many interesting questions have either been answered, or that essential new ideas are needed. In this survey paper, we discuss the most relevant results, open questions, and heuristics.

KEY WORDS: Weakly and strictly self-avoiding walk; polymer measure; large deviation; law of large numbers; central limit theorem; lace expansion.

INTRODUCTION

In the last decades, random polymers received an enormous amount of attention from chemists, physicists and mathematicians. The problems in this field are of practical interest and are, although easy to formulate, rather difficult to solve. Many methods, like computer simulations, heuristics, measurements, mathematical methods, etc., have been applied to answer the main questions. Nevertheless, most of the important problems about two-, three- and four-dimensional random polymers could not be solved rigorously yet. In five or more dimensions, the situation is much better, since starting in 1985 the so-called lace expansion has been turned into a successful tool for the investigation of polymer and other statistical physical models (see also Section 2.4).

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Particularly in one dimension, after some pioneering work in the mid-eighties, decisive progress in the mathematical understanding of random polymers was made during the last decade. Many basic questions concerning scaling behaviour and dependence upon model parameters have been answered. We currently seem to be in the situation that many of the, in our opinion, interesting problems have either been solved, or that the solutions need essential new ingredients. We seem to have reached the limits of what we can prove with the current techniques, and this survey paper is meant to summarize the “state of the art.” We reflect the most relevant results about one-dimensional random polymers, present some heuristic arguments, and formulate a number of conjectures and remaining questions. We also include a heuristic for a conjecture concerning two-dimensional polymers, as the argument is related to a one-dimensional result. The two most successful tools, large deviation techniques and the lace expansion, are reviewed in some detail in Sections 1.3 and 2.4, respectively. We also introduce some models that have not been investigated in the literature, and give conjectures concerning the scaling behavior.

We do not assume any special knowledge other than basic probability theory; this text is self-contained. The paper is organized as follows. In Section 1, we introduce and discuss the basic model, the so-called *weakly self-avoiding walk*. Some more detailed models, including additional features like stiffness, repulsive media or inhomogeneity, are discussed in Section 2. More fancy models and variants, like branching, charged, or hetero-polymers, are presented in Section 3.

We will give a brief overview of the status in other dimensions than one now. We will hardly mention results in dimensions other than one in the rest of the paper.

There has been recent progress in the understanding of two-dimensional self-avoiding walk, using the notion of *conformal invariance*. The values of the critical exponents have been predicted by Nienhuis [Ni84], and these values have been confirmed by Monte Carlo simulations. There is also a mathematical explanation for these remarkable values by Werner, Lawler and Schramm. Roughly speaking, it can be expected that important questions for the scaling behavior of two-dimensional self-avoiding walk could be answered if we would know that self-avoiding walk is conformally invariant. However, even a proper formulation of the latter is not clear (see the review paper [Wer00] and the references therein).

Any understanding of the three-dimensional polymer is lacking, as even the physics literature does not offer any acceptable heuristics. There are even no conjectures what the values of the critical exponents are, even though there are estimates obtained using substantial Monte Carlo simulations.

There is also recent progress in the understanding of four-dimensional (weakly) self-avoiding walk. It is expected that the endpoint, scaled like the free walk, but with logarithmic corrections, tends to a Gaussian. This has been proven by Brydges and Imbrie [BI1, BI2] for the so-called hierarchical lattice.

In high dimensions, the lace expansion has been used to show that (weakly) self-avoiding walk scales as Brownian motion (see [HS94]). For a more detailed explanation of the high-dimensional results, the interested reader is referred to e.g., [BSp85, MS93, HS94], and the references therein. The standard mathematical introduction to random polymers is [MS93], and good introductory texts covering certain aspects of this subject are [dH96, SI96], and [Fr81]. For an introduction to polymer models from a physicist's or chemist's point of view, see [Fl49, vdZ98] and [dG79].

1. WEAKLY SELF-AVOIDING WALK

1.1. Model and Motivation

A random polymer is a long chain of molecules which have the tendency to occupy the space nowhere too tightly resulting in an effective self-repellence. The reason for this self-repellence is the so-called *excluded-volume-effect*: It is energetically favorable for the polymer to spread out over a large area. If S_0, S_1, \dots, S_n denote the locations of the molecules, then this means that there is a penalty for self-intersections, which are index pairs $i \neq j$ such that $S_i = S_j$. One way to describe a polymer is the following model based on a *random walk with self-repellence*. We are going to define the prototype of a *polymer measure*, a probability measure on n -step paths in the space \mathbb{Z}^d .

Fix $n \in \mathbb{N}$ and let $S = (S_0, S_1, \dots, S_n)$ be the first $n+1$ positions of a simple random walker on \mathbb{Z}^d , starting at the origin ($S_0 = 0$). Let P_n be the distribution of S (which is the uniform distribution on the set of all n -step nearest-neighbor paths) and let E_n be expectation w.r.t. P_n . Introducing a parameter $\beta \in [0, \infty)$, we define a new path law $Q_n = Q_n^\beta$ by setting

$$Q_n(S) = \frac{1}{Z_n} e^{-H_n(S)} P_n(S), \quad \text{where } Z_n = E_n e^{-H_n} \quad (1.1)$$

and $H_n = H_n(S)$ is the so-called *Hamiltonian*

$$H_n(S) = \beta \sum_{\substack{i, j=0 \\ i \neq j}}^n \mathbb{1}\{S_i = S_j\} \quad (1.2)$$

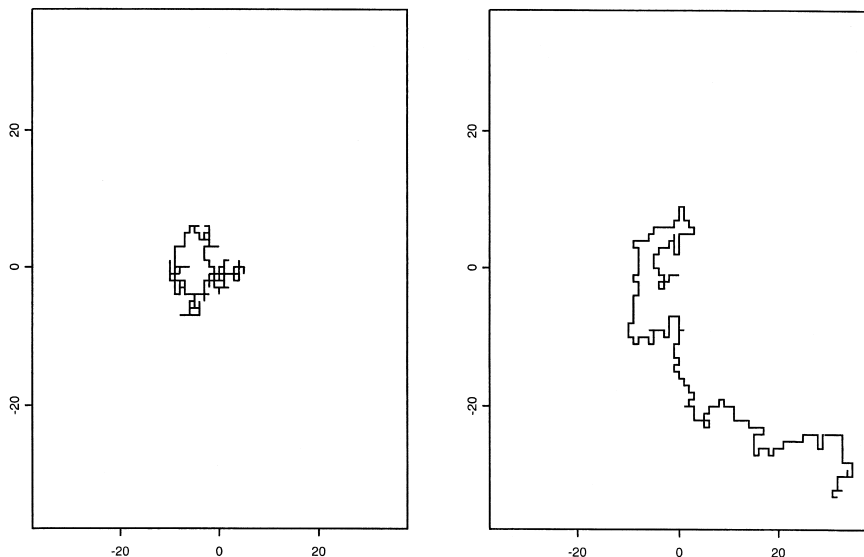


Fig. 1. A typical 200-step simple random walk path, and a typical 200-step weakly self-avoiding walk path with $\beta=1$.

The law Q_n is called the n -polymer measure with strength of self-repulsion β . Equations (1.1)–(1.2) define what is called the Domb–Joyce model for “soft polymers” (see [MS93] Section 10.1). The n -step path receives a penalty $e^{-\beta}$ for every self-intersection. This causes an effective self-repulsion, which tends to spread out the walk. This is clearly demonstrated in a simulation of a two-dimensional simple random walk, respectively, a polymer in Fig. 1. The figure is made using a Monte Carlo simulation. The simple random walk has 414 self-intersections, while the weakly self-avoiding walk only has 18 self-intersections, illustrating the extreme self-repulsion of weakly self-avoiding walk.

The boundary cases $\beta=0$ and $\beta=\infty$ correspond to simple random walk and (strictly) self-avoiding walk, respectively.³ Therefore, the Domb–Joyce model is often referred to as the weakly self-avoiding walk.

This simple model is just a caricature of reality. However, experiments involving real polymers show that in dimension 3, the scale of the weakly self-avoiding walk is very close to the scale of a true polymer in a so-called

³ Indeed, for $\beta=\infty$, the law Q_n is the conditional distribution given that the path S has no self-intersection.

good solvent when the number of building blocks increases. Hence, this model seems to capture the essence of a polymer in a good solvent.⁴

Note that the parameter n is static. The measure Q_n cannot be obtained from Q_{n+1} by projecting onto the first n steps of the path. Indeed, it is difficult to couple polymers of different lengths on one probability space. This corresponds to the idea that growing polymers can completely rearrange themselves in space in order to minimize their energy. Physically, it models the situation in which the thermal motion is much faster than the growth, so that the polymer can rearrange its shape each time it grows.⁵

The double sum in (1.2) is called the *self-intersection local time*, i.e., the time the walker spends at self-intersection points. In terms of the so-called *local times* of the walk,

$$\ell_n(x) = \#\{0 \leq i \leq n : S_i = x\}, \quad n \in \mathbb{N}_0, x \in \mathbb{Z}^d \quad (1.3)$$

H_n can be rewritten as

$$H_n(S) = \beta \sum_{x \in \mathbb{Z}^d} \ell_n^2(x) - \beta(n+1) \quad (1.4)$$

In this way, Q_n arises from the free path measure P_n via a quadratic functional of the local times.

The question that received the most attention is the determination of the large- n behavior of the expected end-to-end distance of the polymer: What is the asymptotics of $E_{Q_n} |S_n|$ as $n \rightarrow \infty$? Deeper questions are the limiting distribution of a properly scaled version of S_n or even of the whole path (S_0, \dots, S_n) in an appropriate sense.

If not stated differently, we will henceforth restrict ourselves to the one-dimensional case, i.e., we put $d = 1$.

1.2. Main Results

It is not difficult to see that the end-to-end distance $|S_n|$ should grow linearly with n under the polymer measure Q_n . (Note that $E_{Q_n}(S_n) = 0$ by the obvious left-right symmetry.) A first result into this direction was derived by Bolthausen [Bo90] who showed that $E_{Q_n} |S_n|/n$ is asymptotically bounded away from zero as $n \rightarrow \infty$ (for a much larger class of random

⁴ In terms of the critical exponent ν defined in (1.24) below, we see that the experimental value obtained from neutron diffraction studies is $\nu = 0.586 \pm 0.004$, whereas the Monte Carlo value for the critical exponent ν for the tree-dimensional self-avoiding walk is $\nu = 0.588 \pm 0.0015$ (see [vdZ98] and the references therein).

⁵ A model, where the chain evolves randomly as the length parameter increases without being able to rearrange itself is discussed in Section 3.4 below.

walks than only nearest neighbour in Section 1.1). At first sight surprisingly, this result could only be proved for sufficiently small β . However, later it was realized that the method used in [Bo90] is essentially an expansion around the Edwards model (see Section 1.4) and is therefore naturally restricted to small β (see also Theorems 1.3–1.5 below).

The main result for the one-dimensional weakly self-avoiding walk is the following central limit theorem. By \mathcal{N} we denote the standard normal distribution.

Theorem 1.1. For every $\beta \in (0, \infty)$, there exists $\theta^* = \theta^*(\beta) \in (0, 1)$ and $\sigma^* = \sigma^*(\beta) \in (0, \infty)$ such that

$$\lim_{n \rightarrow \infty} Q_n \left(\frac{|S_n| - \theta^* n}{\sigma^* \sqrt{n}} \leq C \right) = \mathcal{N}((-\infty, C]) \quad \text{for every } C \in \bar{\mathbb{R}} \quad (1.5)$$

Moreover,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log Z_n = r^*(\beta) \quad (1.6)$$

The quantity θ^* is called the *speed* of the polymer, whereas σ^* is called its *spread*. In physicists' language, Theorem 1.1 says that the weakly self-avoiding walk behaves ballistically with normal fluctuations around the linear drift. The law of large numbers contained in Theorem 1.1 was proved by Greven and den Hollander [GH93] using a Markovian description of the local times in (1.3), combined with large-deviation arguments and a spectral analysis. The central limit theorem was proved by König [Kö96] via a construction of a transformed Markov chain. We will give a sketch of these proofs in Section 1.3 below.

The rate function

$$I(\theta) = -\lim_{n \rightarrow \infty} \frac{1}{n} \log Q_n(S_n \approx \theta n) \quad (1.7)$$

is symmetric, and it has been shown in [Kö94] that I is real-analytic and strictly convex in a neighborhood of θ^* and that the only zeros of I are $\pm \theta^*$. The latter assertion instantly implies the law of large numbers in Theorem 1.1, and the fact that $I''(\theta^*) > 0$ suggests the validity of the central limit theorem in Theorem 1.1 with variance $(\sigma^*)^2 = I''(\theta^*)^{-1}$. Pushing the arguments of [Kö94] forward, den Hollander [dH00] shows that I is differentiable and convex in $(0, 1)$ and is linear for θ close to zero.

In [GH93] it is proved that $\beta \mapsto \theta^*(\beta)$ is a real-analytic function satisfying $\lim_{\beta \downarrow 0} \theta^*(\beta) = 0$ and $\lim_{\beta \rightarrow \infty} \theta^*(\beta) = 1$. The following conjecture is appealing:

Conjecture 1.2. The map $\beta \mapsto \theta^*(\beta)$ is strictly increasing.

Indeed, if the penalty for self-intersections is stronger, then we expect the walk to spread out more. Unfortunately, we have no hope for proving Conjecture 1.2 via a coupling argument since it is very difficult to couple Q_n^β and $Q_n^{\beta'}$ for $\beta \neq \beta'$. Also the functional analytic description of $\theta^*(\beta)$ obtained from the proof of Theorem 1.1 (see Section 1.3) is not explicit enough to allow for a proof of Conjecture 1.2.

We have no doubt that also an invariance principle is true, i.e., the scaled path $((\sigma^*)^2 n)^{-1/2} (S_{\lfloor nt \rfloor} - \theta^* nt)_{t \in [0, 1]}$ converges weakly towards standard Brownian motion. Tightness of the law of this process is an easy consequence of the proof of Theorem 1.1, together with good bounds on the normalizing constant Z_n .⁶ A proof for the convergence of the finite dimensional distributions should follow from an obvious extension of the method used in the proof of Theorem 1.1. However, the details have not been carried out.

There is an intricate scaling behaviour as $\beta \downarrow 0$. The following theorem is taken from [HH95] and [HHK97b]:

Theorem 1.3. There exist $a^*, b^*, c^* \in (0, \infty)$ such that, as $\beta \downarrow 0$,

$$\beta^{-2/3} r^*(\beta) \rightarrow a^*, \quad \beta^{-1/3} \theta^*(\beta) \rightarrow b^*, \quad \sigma^*(\beta) \rightarrow c^* \quad (1.8)$$

See the end of Section 1.3 for some explanation of the proof. In Section 1.5, a sketch of a proof of the first two assertions is given. Here we first give an *intuitive* argument for the first two scaling assertions in (1.8). Assume that $S_n \approx \theta n$ (meaning that $S_n = \lfloor \theta n \rfloor$ or $S_n = \lfloor \theta n \rfloor + 1$), and that the path S spends all its time in between 0 and S_n . In this case, the minimal value of the Hamiltonian in (1.4) is $\frac{\beta}{\theta} n$. Indeed, the Hamiltonian is β times the sum of squares of the local times, and the sum of the local times is equal to n . Hence, the sum of squares is minimal when all local times are equal to $\frac{1}{\theta}$. In this case, $H_n \approx \frac{\beta}{\theta} n$. The probability for $S_n \approx \theta n$ is, for small θ , approximately $e^{-n\theta^2/2}$. In total, this gives that the contribution to the normalizing constant of paths with $S_n \approx \theta n$ is roughly $e^{-n[\beta/\theta + \theta^2/2]}$. Maximizing

⁶ Indeed, tightness for self-repellent models (see [MS93] Chapter 6) is implied by the two conditions (1) $E_{Q_n}(S_n - \theta^* n)^4 = \mathcal{O}(n^2)$ and (2) the existence of $\lim_{n \rightarrow \infty} e^{nr^*} Z_n$ for some $r^* \in \mathbb{R}$. Both properties follow from the proof of Theorem 1.1 for the one-dimensional weakly self-avoiding walk.

the exponential rate over θ gives that the speed behaves as $\theta^*(\beta) \sim \beta^{1/3}$, and the rate of the normalizing constant as $r^*(\beta) \sim \beta^{2/3}$. Here we use \sim to denote equivalence up to constants.

Rigorous bounds on a^* , b^* , and c^* appeared in [vdH98b]. The most important bound is that $c^* < 1$, which shows that there is a discontinuity of the variance $\sigma^*(\beta)$ at $\beta=0$, since $\sigma^*(0)=1$ for simple random walk. The fact that $\lim_{\beta \downarrow 0} \sigma^*(\beta) < 1$ is deeper than the above simplified argument. It is related to the analysis of the Edwards model, the continuous space/time analogue of the weakly self-avoiding walk based on Brownian motion (see Section 1.4).

The following result, taken from [HHK97b], deals with the case where the self-repulsion strength $\beta = \beta_n$ depends on n and tends to zero as $n \rightarrow \infty$. It is clear that the result must be trivial if β_n vanishes too fast. The borderline is marked by the choice $\beta_n = n^{-3/2}$, where we have Brownian scaling by the invariance principle (see also [BSI95] and Section 1.4 below).

Theorem 1.4. If β is replaced by $\beta_n \rightarrow 0$ and $\beta_n n^{3/2} \rightarrow \infty$, then the central limit theorem in (1.5) persists with $\theta^* = \theta^*(\beta_n) = b^* \beta_n^{1/3} (1 + o(1))$ and $\sigma^* = c^*$.

The message of Theorem 1.4 is that the limits $n \rightarrow \infty$ in (1.5) and $\beta \downarrow 0$ in (1.8) are *uniform*, so that they can be coupled by substituting $\theta^* = \theta^*(\beta_n)$ and $\sigma^* = \sigma^*(\beta_n)$. The proof of Theorem 1.4 is a refinement of the proof of Theorem 1.1 and is technically much more involved. A simpler proof of the law of large numbers in Theorem 1.4 based merely on the weak convergence of simple random walk to Brownian motion is presented in Section 1.5.

1.3. Sketch of Proofs

The proof of Theorem 1.1 is based on two main ingredients: (1) a Markovian representation for the joint description of the local times $(\ell_n(x))_{x \in \mathbb{Z}}$ in (1.3) together with the endpoint S_n , and (2) a transformation of the Markov chain involved to incorporate the factor e^{-H_n} . The first idea has been successfully adapted to prove similar results for other one-dimensional path models. The second idea is a variant of a well-known technique widely used in large-deviation theory (see e.g., the proof of Cramér's theorem). In the following, we outline the proof of Theorem 1.1. A more thorough and extensive account of this proof is provided in [dH00].

We want to analyze the quantity $E_n(e^{-H_n} \mathbb{1}\{S_n \approx \theta n\})$ for $\theta \in (0, 1)$, where again \approx means that $S_n = \lfloor \theta n \rfloor$ or $S_n = \lfloor \theta n \rfloor + 1$. As a simplification of

the problem, we assume that the path spends all its time until n in the interval $[0, \theta n]$. This assumption can be justified for all $\theta \in (\theta^{**}, 1)$ for some $\theta^{**} \in (0, \theta^*)$.⁷ Certainly, in order to derive the full statement of Theorem 1.1, one has to deal carefully with the pieces of the path that lie outside the interval $[0, \theta n]$, but let us neglect them for this outline. Hence, we want to analyze the quantity $Z_n(\theta)$ defined by

$$Z_n(\theta) = E_n \left(\exp \left[-\beta \sum_{0 < x \leq \theta n} \ell_n(x)^2 \right] \mathbb{1}_{\{S_n \approx \theta n\}} \mathbb{1}_{\left\{ \sum_{0 < x \leq \theta n} \ell_n(x) = n \right\}} \right) \quad (1.9)$$

The first main idea is that the local times are a simple functional of a Markov chain. Indeed, let $m_n(x)$ be the number of up-crossings of the path S from x to $x+1$, i.e., the number of steps from $x \rightarrow x+1$. Then, on the set $\{S_n \approx \theta n, \sum_{0 < x \leq \theta n} \ell_n(x) = n\}$, we have $\ell_n(x) = m(x-1) + m(x) - 1$ for all $0 < x \leq \theta n$. Furthermore, as is proven by Knight [Kn63], the sequence $(m_n(x))_{x=0, \dots, \lfloor \theta n \rfloor}$ is a homogeneous Markov chain on \mathbb{N} . More precisely, it is a critical Galton–Watson branching process with geometric offspring distribution and one immigrant per generation.⁸ In particular, this chain is null-recurrent. Its transition kernel is given by

$$P(i, j) = \binom{i+j-2}{i-1} \left(\frac{1}{2}\right)^{i+j-1}, \quad i, j \in \mathbb{N} \quad (1.10)$$

Hence, if E denotes the expectation with respect to a Markov chain $(m_x)_{x \in \mathbb{N}_0}$ with kernel P in (1.10), then we obtain that

$$Z_n(\theta) \approx E \left(\exp \left[-\beta \sum_{0 < x \leq \theta n} (m_{x-1} + m_x - 1)^2 \right] \times \mathbb{1}_{\left\{ \sum_{0 < x \leq \theta n} (m_{x-1} + m_x - 1) = n \right\}} \right) \quad (1.11)$$

The second main step is a change to a transformed Markov chain such that the quadratic functional is absorbed into the transition mechanism of the new chain. For this purpose, define, for any $r \in \mathbb{R}$, an infinite matrix $A_{r, \beta}$ by

$$A_{r, \beta}(i, j) = e^{r(i+j-1) - \beta(i+j-1)^2} P(i, j), \quad i, j \in \mathbb{N}$$

⁷ It turns out (see [dH00]) that, for $\theta \in (0, \theta^{**})$, it is cheaper to spend a large fraction of time outside $[0, \lfloor \theta n \rfloor]$. This leads to the fact that the rate function $\theta \mapsto I(\theta)$ is linear for $\theta \in [0, \theta^{**}]$.
⁸ In fact, Knight used this observation as a main tool to establish the analogous continuous assertion, the so-called Ray–Knight description of the Brownian local times.

Note that $A_{r,\beta}$ defines a symmetric, compact and componentwise positive operator $\ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$. Hence, it has a largest eigenvalue $\lambda(r, \beta)$ with corresponding unique ℓ^2 -normalized positive eigenvector $\tau_r = (\tau_r(i))_{i \in \mathbb{N}}$. We define a stochastic matrix $P_{r,\beta}$ by

$$P_{r,\beta}(i, j) = \frac{A_{r,\beta}(i, j) \tau_r(j)}{\lambda(r, \beta) \tau_r(i)}, \quad i, j \in \mathbb{N}$$

Denote the law of a Markov chain $(m_x)_{x \in \mathbb{N}_0}$ with kernel $P_{r,\beta}$ by $\hat{P}^{r,\beta}$. Observe that $e^{rn} = \prod_{0 < x \leq \theta n} e^{r(m_{x-1} + m_x - 1)}$ on the r.h.s. of (1.11). Hence, the change of the Markovian measure yields

$$Z_n(\theta) \approx e^{-rn\lambda(r, \beta)} \theta^n \hat{P}^{r,\beta} \left(\sum_{0 < x \leq \theta n} (m_{x-1} + m_x - 1) = n \right) \quad (1.12)$$

(Note that the factors τ_r are telescoping.) The Markov chain has good recurrence properties under $\hat{P}^{r,\beta}$. In particular, τ_r^2 turns out to be the invariant measure and $m_0 + m_1 - 1$ has expectation $\partial_r \log \lambda(r, \beta)$ when the chain is started in this distribution. Equation (1.12) is true for any $r \in \mathbb{R}$. Now we pick $r = r(\theta)$ such that the probability $\hat{P}^{r,\beta}(\sum_{0 < x \leq \theta n} (m_{x-1} + m_x - 1) = n)$ does not decay exponentially fast. By the law of large numbers for ergodic Markov chains and the fact that $m_x + m_{x-1} - 1$ has expectation $\partial_r \log \lambda(r, \beta)$ when the chain is started in the invariant distribution, this condition means that $\partial_r \log \lambda(r, \beta)|_{r=r(\theta)} = \frac{1}{\theta}$. This identifies the logarithmic asymptotics as

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\theta) = \theta \log \lambda(r(\theta), \beta) - r(\theta) \quad (1.13)$$

Maximizing over $\theta \in (0, 1)$ and using that $\partial_r \log \lambda(r, \beta)|_{r=r(\theta)} = \frac{1}{\theta}$ yields that the maximal value $\theta = \theta^*$ is characterized by the two equations

$$\lambda(r^*, \beta) = 1 \quad \text{and} \quad \frac{1}{\theta^*} = \partial_r \log \lambda(r, \beta)|_{r=r^*} \quad (1.14)$$

In particular, we also obtain the exponential rate of the normalizing constant as

$$r^* = r(\theta^*) = - \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n = - \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\theta^*) \quad (1.15)$$

Equations (1.13) and (1.15) yield the following analytic expression for the rate function I in (1.7):

$$I(\theta) = r^* - \theta \log \lambda(r(\theta), \beta) + r(\theta), \quad \theta \in (\theta^{**}, 1) \tag{1.16}$$

For a proof of Theorem 1.1, one has to show that I is bounded away from zero outside any neighborhood of $\pm \theta^*$ and must fill in the gaps in the analysis described so far for $\theta = \theta^*$.

The proof of Theorem 1.3 follows from a scaling analysis for the eigenvalue $\lambda(r, \beta)$ as $r, \beta \rightarrow 0$. In fact, it is proved in [HH95] that, for any $a \in \mathbb{R}$, we have $\lambda(a\beta^{2/3}, \beta) = 1 + \beta^{1/3}(\varrho(a) + o(1))$ as $\beta \downarrow 0$, where $\varrho(a)$ is the principal eigenvalue of a certain second-order differential operator on $L^2([0, \infty))$. (This operator plays an analogous role for the Edwards model (see Section 1.4) as the matrix $A_{r, \beta}$ plays for the weakly-self-avoiding walk.) The proof involves the notion of epi-convergence in terms of which the convergence of the variational problem for $\lambda(a\beta^{2/3}, \beta)$ towards the one for $\varrho(a)$ is formulated. The constants a^* and b^* are characterized by

$$\varrho(a^*) = 0 \quad \text{and} \quad \varrho'(a^*) = \frac{1}{b^*} \tag{1.17}$$

1.4. Relation to the Edwards Model

In this section, we define a continuous space/time analog to the weakly self-avoiding walk, the so-called *Edwards model*.

Let $B = (B_t)_{t \geq 0}$ be standard Brownian motion on \mathbb{R} , starting at the origin. For $T > 0$ and a parameter $\beta \in (0, \infty)$, formally define a path measure $\hat{Q}_T = \hat{Q}_T^\beta$ by the Radon–Nikodym derivative

$$\frac{d\hat{Q}_T}{d\hat{P}}(dB) = \frac{1}{\hat{Z}_T} \exp \left[-\beta \int_0^T ds \int_0^T dt \delta(B_s - B_t) \right] \tag{1.18}$$

where \hat{Z}_T is the appropriate normalizing constant and \hat{P} is the Wiener measure. The Edwards model at time $T=1$ appears as the natural scaling limit for the weakly self-avoiding walk if the self-repellence strength $\beta = \beta_n$ is chosen as $\beta_n = \text{const. } n^{-3/2}$; see [BSI95]. The law \hat{Q}_T is called the *T-polymer measure with strength of repulsion β* . As in (1.4), the double integral in (1.18), also called the *intersection local time*, is rigorously defined in terms of the Brownian local times $(L(T, x))_{x \in \mathbb{R}}$ as

$$\int_0^T ds \int_0^T dt \delta(B_s - B_t) = \int_{\mathbb{R}} L(T, x)^2 dx \tag{1.19}$$

The following result is analogous to Theorem 1.1. The constants b^* and c^* were introduced in Theorem 1.3.

Theorem 1.5. The central limit theorem in (1.5) but now for $(\hat{Q}_T)_{T>0}$ holds for every $\beta \in (0, \infty)$ if n is replaced by T , $\theta^*(\beta)$ by $b^*\beta^{1/3}$, and $\sigma^*(\beta)$ by c^* .

The law of large numbers in Theorem 1.5 was first proved in [We84], which was the first rigorous result for a one-dimensional polymer model, up to our best knowledge (in [Ku84] a weaker version was proven). The central limit theorem was proved in [HHK97a], using the continuous version of the proof sketched in Section 1.3. Indeed, the Ray–Knight theorems for the Brownian local times are combined with a Girsanov transformation for the squared Bessel process involved. The objects defined in the proof of Theorem 1.4 in [HHK97b], are discrete approximations of the corresponding continuous objects appearing in the proof of Theorem 1.5 in [HHK97a]. One can consider $Q_n^{\beta_n}$ with $\beta_n \downarrow 0$ and $\beta_n n^{3/2} \rightarrow \infty$ as a discrete approximation to $\hat{Q}_T^{\beta_T}$ with $\beta_T \rightarrow \infty$ as $T \rightarrow \infty$.

It is noteworthy that the speed $b^*\beta^{1/3}$ in Theorem 1.5 depends on β in such a simple manner and that the spread c^* does not depend on β at all. These facts are direct consequences of the Brownian scaling property, as can be seen from an elementary calculation. Indeed, the law of B_T under \hat{Q}_T^β is the same as the law of $\beta^{-1/3}B_{\beta^{2/3}T}$ under $\hat{Q}_{\beta^{2/3}T}^1$. Recall that the spread c^* of the Edwards model is *smaller* than the spread of the free Brownian motion, see [vdH98b]. Apparently, the self-repellence reduces fluctuations, but we have no intuitive explanation for this phenomenon.

1.5. Weak-Interaction Limits

The results in the preceding subsections suggest that the Edwards model is, in some sense, the weak-interaction limit of discrete polymer models as the length n tends to infinity and the interaction strength β tends to zero afterwards. This will be demonstrated by the following intuitive calculation. It is possible to turn the argument below into a proper proof of the first two scaling assertions in Theorem 1.3 using detailed knowledge about the Edwards model as obtained in the proof of Theorem 1.5. This idea and a possible extension to proofs of universality assertions and of Conjecture 2.3 below are in [HHK01].

The argument is roughly as follows. For $\mu \in \mathbb{R}$ consider

$$Z_n^\beta(\mu) = E_n(e^{-H_n + \mu\beta^{1/3}S_n}) \quad (1.20)$$

Fix a large $T > 0$ and assume that β is small. Splitting the polymer (S_0, \dots, S_n) into $n\beta^{2/3}/T$ pieces of equal length and neglecting the interaction between different pieces, we obtain the bound

$$Z_n^\beta(\mu) \leq (Z_{T\beta^{-2/3}}^\beta(\mu))^{n\beta^{2/3}/T} \tag{1.21}$$

Next we use the weak convergence as $\beta \downarrow 0$ of the scaled walk $(\beta^{1/3}S_{t\beta^{-2/3}})_{t \in [0, T]}$ towards standard Brownian motion $(B_t)_{t \in [0, T]}$ to see that

$$\lim_{\beta \downarrow 0} Z_{T\beta^{-2/3}}^\beta(\mu) = \hat{Z}_T(\mu) = \hat{E} \left(\exp \left\{ - \int L(T, x)^2 dx + \mu B_T \right\} \right) \tag{1.22}$$

Using this in (1.21), we see that

$$\limsup_{\beta \downarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{\beta^{2/3}n} \log Z_n^\beta(\mu) \leq \frac{1}{T} \log \hat{Z}_T(\mu) \tag{1.23}$$

From the proof of Theorem 1.5 in [HHK97a] one obtains an analytic description of the limit of the r.h.s. as $T \rightarrow \infty$ in terms of the function g mentioned at the end of Section 1.3. In particular, one can easily derive (via the exponential Chebyshev inequality and the Laplace method) that, for any $b > b^*$, $Q_n^\beta(S_n > b\beta^{1/3}n)$ vanishes as $n \rightarrow \infty$ and $\beta \downarrow 0$. The derivation of the lower bound of the normalizing constant is technically more involved and needs more detailed information of the Edwards model.

1.6. Heuristics for $d=2$

In this section, we will use Theorem 1.4 above to give a heuristic argument showing that the size of the self-avoiding walk in dimension $d=2$ is $n^{3/4}$. This argument is an adaptation of the well-known Flory argument (see [MS93] Section 2.2). The Flory argument gives heuristic values for the critical exponent

$$v = v(d) = \lim_{n \rightarrow \infty} \frac{1}{\log n} \log E_{Q_n} |S_n| \tag{1.24}$$

These values are $v_F(2) = \frac{3}{4}$, $v_F(3) = \frac{3}{5}$, $v_F(d) = \frac{1}{2}$ for $d \geq 4$ and are consistent with Monte Carlo simulations except in dimension 3, where computer simulations suggest that rather $v(3) = 0.588\dots$ However, the original Flory argument is extremely rough, and it is completely unclear to us why it is so accurate.

We will give a refinement of this argument that relates two-dimensional strictly self-avoiding walk to one-dimensional weakly self-avoiding walk

with an appropriate choice of $\beta = \beta_n$. This argument can be extended to higher dimensions. However, it becomes worse in higher dimension, and in fact, in general dimension precisely gives the Flory exponents described above.

Write $S_i = (S_i^{(1)}, S_i^{(2)})$ for the two components. We may assume that $S^{(1)} = (S_0^{(1)}, \dots, S_n^{(1)})$ and $S^{(2)} = (S_0^{(2)}, \dots, S_n^{(2)})$ are two independent one-dimensional n -step simple random walks.⁹ We want to make a connection between

$$Z_n^{(\nu)} = P \left(\bigcap_{\substack{i,j=0 \\ i \neq j}}^n \{S_i \neq S_j\} \cap \{|S_n| = \mathcal{O}(n^\nu)\} \right) \tag{1.25}$$

and the weakly self-avoiding walk model for $S^{(1)}$ with parameter $\beta = n^{-\nu}$. Later we derive the appropriate value of $\nu \in [\frac{1}{2}, 1]$.

Let $E^{(1)}$ be expectation w.r.t. $S^{(1)}$ and rewrite

$$Z_n^{(\nu)} \approx E^{(1)} \left(\mathbb{1}_{\{|S_n^{(1)}| = \mathcal{O}(n^\nu)\}} P \left(\bigcap_{\substack{i,j=0 \\ i \neq j}}^n \{S_i \neq S_j\} \cap \{|S_n^{(2)}| = \mathcal{O}(n^\nu)\} \mid S^{(1)} \right) \right) \tag{1.26}$$

Denote the local times of $S^{(1)}$ by $\ell_n(x)$. Recall that $S^{(1)}$ has precisely $\ell_n(x)$ ($\ell_n(x) - 1$) self-intersections at $x \in \mathbb{Z}$. In order that S has no self-intersections, $S^{(2)}$ must avoid self-intersections at precisely those $\sum_{x \in \mathbb{Z}} \ell_n(x)(\ell_n(x) - 1)$ time pairs at which $S^{(1)}$ has self-intersections. Now we make the two caricature assumptions that (1) self-intersections occur independently, and that (2) the probability for a self-intersection of $S^{(2)}$ at a given time pair $i \neq j$ at which $S_i^{(1)} = S_j^{(1)}$ is roughly equal to $n^{-\nu}$. (The idea behind (2) is that $|i - j|$ is typically large and that for $|i - j|$ large $S_i^{(2)} - S_j^{(2)}$ is more or less uniformly distributed on $\{-\lfloor n^\nu/2 \rfloor, \dots, \lfloor n^\nu/2 \rfloor\}$.) Hence, we have roughly

$$P \left(\bigcap_{\substack{i,j=0 \\ i \neq j}}^n \{S_i \neq S_j\} \cap \{|S_n^{(2)}| = \mathcal{O}(n^\nu)\} \mid S^{(1)} \right) \approx \prod_{x \in \mathbb{Z}} (1 - n^{-\nu})^{\ell_n(x)(\ell_n(x) - 1)} \\ \approx \exp \left[-n^{-\nu} \sum_{x \in \mathbb{Z}} \ell_n(x)^2 \right]$$

Using this approximation in (1.25), we have an approximation for $Z_n^{(\nu)}$ as the partition sum of the Domb–Joyce model for $S^{(1)}$ with parameter $\beta = n^{-\nu}$ and the additional indicator on the event $\{|S_n^{(1)}| = \mathcal{O}(n^\nu)\}$.

⁹ In fact, the projections of $S^{(1)}$ and $S^{(2)}$ onto the lines with slope 1 and -1 , respectively, are two i.i.d. copies of one-dimensional simple random walks.

We know from Theorem 1.4 that the maximal contribution to $\exp[-n^{-\nu} \sum_x \ell_n(x)^2]$ comes from $\{|S_n^{(1)}| = \mathcal{O}(n^{1-\nu/3})\}$. Equating the optimal power $1 - \frac{\nu}{3}$ with the power ν under consideration, we obtain that ν should be equal to $\frac{3}{4}$.

The above argument is quite crude. For instance, it predicts that the normalizing constant behaves as $e^{-\mathcal{O}(\sqrt{n})}$, whereas we know that it is $e^{-\mathcal{O}(n)}$. This is due to the fact that the approximation is bad for i and j that are close together, i.e., for short loops. The original argument by Flory was even cruder. He assumed that both $S_1^{(1)}, \dots, S_n^{(1)}$ and $S_1^{(2)}, \dots, S_n^{(2)}$ are i.i.d. uniform on $\{-\lfloor n^\nu/2 \rfloor, \dots, \lfloor n^\nu/2 \rfloor\}$, and compared the probability for S to have no self-intersections to the probability that $S_n = \mathcal{O}(n^\nu)$. The Flory argument was strongly criticized by des Cloiseaux [Cl75]. However, a more careful analysis, using a mean-field approximation, in fact yields the worse result $\nu(3) = 2/3$. See [CPP94] for an explanation of the heuristic method of des Cloiseaux.

Our neglect of the short loops in the heuristics above is considered harmless, since it is the suppression of *long* loops that dictates the scale of the polymer. Indeed, if we would only incorporate loops of length less than or equal to m in the Hamiltonian, then we would only get the self-avoiding walk scaling when $m = m_n \rightarrow \infty$ sufficiently fast. It is an interesting problem to determine *how fast* m_n should go to infinity for the scales of the two models to be identical.

2. MORE REFINED MODELS

The weakly self-avoiding walk is a caricature of reality. In this section we discuss some models that are more refined.

2.1. Repulsion and Attraction

This time, with two parameters $\beta, \gamma \in (0, \infty)$, we choose the Hamiltonian to be

$$H_n(S) = \beta \sum_{\substack{i, j=0 \\ i \neq j}}^n \mathbb{1}\{S_i = S_j\} - \frac{\gamma}{2d} \sum_{i, j=0}^n \mathbb{1}\{|S_i - S_j| = 1\} \quad (2.1)$$

and define a polymer measure $Q_n = Q_n^{\beta, \gamma}$ by (1.1). This is a model for a polymer in a repulsive solution. The first term in (2.1) is again the self-intersection local time and models the excluded-volume-effect. The second term, the number of self-contacts, models the fact that the polymer wants to be close to itself because of the repulsion of the solution.

The following theorem in [HK00] describes the qualitative behaviour for different values of the parameters β and γ in any dimension. In its statement, $\|\cdot\|$ is any norm on \mathbb{Z}^d .

Theorem 2.1. Let $d \in \mathbb{N}$ be arbitrary.

(i) For $\gamma < \beta$ there exist $\varepsilon = \varepsilon(\gamma, \beta, d) > 0$ and $c = c(\beta, \gamma, d) > 0$ such that, for all $n \in \mathbb{N}$,

$$Q_n^{\beta, \gamma} \left(\max_{i=0}^n \|S_i\| \leq \varepsilon n^{1/d} \right) \leq e^{-cn} \quad (2.2)$$

(ii) For $\gamma > \beta$, there exists a constant $c = c(\beta, \gamma, d) > 0$ such that for sufficiently large L and all $n \in \mathbb{N}$,

$$Q_n^{\beta, \gamma} \left(\max_{i=0}^n \|S_i\| > L \right) \leq e^{-cLn} \quad (2.3)$$

The above theorem means that there is a phase transition between collapse and infinite size at $\gamma = \beta$. On the level of partition sums, this is reflected by the facts used to prove Theorem 2.1 that $\limsup_{n \rightarrow \infty} \frac{1}{n} \log Z_n \in (-\infty, 0)$ for $\gamma < \beta$ and $\liminf_{n \rightarrow \infty} \frac{1}{n} \log Z_n > 0$ for $\gamma > \beta$.

Specializing Theorem 2.1 to $d=1$, we see that for $\gamma < \beta$ the range of the walk,

$$R_n = \#\{S_0, \dots, S_n\} \quad (2.4)$$

is of the order n , whereas for $\gamma > \beta$ it remains finite.¹⁰ For $d=1$ and all $\gamma < \beta$, it is expected that the statement of Theorem 1.1 holds. However, van der Hofstad and Klenke [HK00] are only able to prove this for $\gamma < \beta - \frac{1}{2} \log 2$.

The critical case $\gamma = \beta$ is investigated in [HKK01] in $d=1$. It turns out that, under $Q_n^{\beta, \beta}$, the endpoint S_n is of the order

$$\alpha_n = \frac{n^{1/2}}{(\log n)^{1/4}} \quad (2.5)$$

¹⁰ For $d \geq 2$, the situation is more complicated, and it is believed that a *second* critical curve $\beta \mapsto \gamma^*(\beta)$ exists with $0 < \gamma^*(\beta) < \beta$. For $\gamma < \gamma^*(\beta)$, the behaviour of the polymer is expected to be the same as for the weakly self-avoiding walk ($\gamma=0$), whereas for $\gamma^*(\beta) < \gamma < \beta$, the endpoint should be of order $n^{1/d}$.

Furthermore, the accordingly scaled local times in (1.3) converge weakly to some explicitly known function, characterized in terms of a variational problem, whose support has size $2(\frac{9}{2}\beta)^{1/4}$. The main result in [HKK01] identifies the limiting behavior of the range R_n in (2.4):

Theorem 2.2. Fix $\beta > 0$ and put $d = 1$. Then, for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} Q_n^{\beta, \beta} \left(\left| \frac{R_n}{\alpha_n} - 2 \left(\frac{9}{2} \beta \right)^{1/4} \right| > \varepsilon \right) = 0 \tag{2.6}$$

2.2. Inhomogeneity

Instead of changing the interaction, one can also change the reference measure in the definition of the polymer model. Let $(S_i)_{i \in \mathbb{N}_0}$ be some random walk (starting at the origin) whose steps are symmetrically distributed on $\{\pm 1, \dots, \pm r\}$ for some $r \in \mathbb{N}$. The polymer measure Q_n is again defined by (1.1), where now P_n denotes the distribution of the path $S = (S_0, \dots, S_n)$. This is a model for an inhomogeneous polymer, which contains molecules with different chemical properties such that the chemical bonds between them are different. Note that Q_n is now non-trivial and interesting also in $d = 1$ for $\beta = \infty$ if $r > 1$, which is the strictly self-avoiding case.

König proves the statement of Theorem 1.1 (see [Kö93, Kö94] and [Kö96]), including the self-avoiding case $\beta = \infty$. Furthermore, he proves that there exists a speed $\theta^* = \theta^*(\beta, r) \in (0, r)$ which converges, as $\beta \rightarrow \infty$, towards the speed $\theta(\infty, r) \in (0, r)$ of the self-avoiding walk. The proof is in the spirit of the proof of Theorems 1.1, but the Markovian description of the local times is much more complicated and less explicit, due to the fact that the walker is allowed to make longer steps.

Aldous [Al86] conjectures the following. (The constant b^* appearing below was introduced in Theorem 1.3.¹¹)

Conjecture 2.3. $\lim_{r \rightarrow \infty} r^{-2/3} \theta^*(\infty, r) = b^* 3^{-1/3}$.

Aldous [Al86] proves a limit law for the appropriately scaled sequence of self-intersection times of the free random walk $(S_i)_{i \in \mathbb{N}_0}$ (i.e., the subsequent random times at which self-intersections occur) as the range r tends to infinity. Conjecture 2.3 is based on interchanging this limit with the limit $n \rightarrow \infty$ and using Theorem 1.5. In [HHK01], the applicability of the idea outlined in Section 1.5 to a proof of Conjecture 2.3 is investigated.

¹¹ In fact, in [Al86] the factor b^* is missing.

Interestingly, via the results of [Al86] one could obtain the Edwards model as a scaling limit of the inhomogeneous weakly self-avoiding walk as the step range r tends to infinity.

It appears to be rather difficult to investigate the weakly self-avoiding walk for a random walk whose steps are unbounded and lie in the domain of attraction of a stable random variable other than a Gaussian. This model would have some flavor of the two-dimensional nearest-neighbor weakly self-avoiding walk.

2.3. Two-Dimensional Strip

In this example, we change the reference measure in (1.1) in the following way: We do not use simple random walk on \mathbb{Z} , but on the two-dimensional strip $\{-r, \dots, r\} \times \mathbb{Z}$ for some $r \in \mathbb{N}$. This is a model for a two-dimensional polymer which is forced to stay between two fixed bounds in one space direction. The constraint makes it essentially a one-dimensional object. Indeed, this model turns out to be much closer to the one-dimensional inhomogeneous model in Section 2.2 than to the two-dimensional self-avoiding walk.

Write $S_i = (S_i^{(1)}, S_i^{(2)})$ for the two components. It would not be difficult to formally adapt the proofs of [Kö93, Kö94] and [Kö96] to obtain a central limit theorem for $S_n^{(2)}$ analogous to Theorem 1.1 (and an additional limit law for $S_n^{(1)}$ on $\{-r, \dots, r\}$ as well), although the precise formulas may be difficult to tract. The proof of the law of large numbers, at least in the self-avoiding case $\beta = \infty$, has indeed been carried out by Alm and Janson [AJ90], in fact for more general lattices than the strip $\{-r, \dots, r\} \times \mathbb{Z}$. Their proof is in the spirit of the proof of [Kö93], but uses a different language.

Unfortunately, all attempts have failed so far to say anything non-trivial about the asymptotics of the speed of the second component of the polymer as the strip width r tends to infinity. The conjecture is that the speed $\lim_{n \rightarrow \infty} E_{Q_n} |S_n^{(2)}|/n$ behaves like $r^{-1/3}$. This result would be a major step towards the solution of the famous open problem of identifying the scaling for the two-dimensional self-avoiding walk. Indeed, under Q_n , we would then have $|S_n^{(2)}| \approx nr^{-1/3}$ and $|S_n^{(1)}| \approx r$. Assuming homogeneity and equating the two, we find that the appropriate scale $r = r_n$ for the two-dimensional self-avoiding walk satisfies $r = nr^{-1/3}$, or $r = n^{3/4}$. However, a justification of the coupling of the limits $n \rightarrow \infty$ and $r \rightarrow \infty$ seems to be difficult.

2.4. Short-Range Interaction

Fix parameters $\beta \in (0, \infty)$ and $p \in \mathbb{R}$. For a path $S = (S_0, \dots, S_n)$, define the Hamiltonian by

$$H_n(S) = \beta \sum_{\substack{i, j=0 \\ i \neq j}}^n \frac{\mathbb{1}\{S_i = S_j\}}{|i - j|^p} \quad (2.7)$$

and define the measure $Q_n = Q_n^{\beta, p}$ as in (1.1). This is a model in which, for $p > 0$, long loops are punished less strongly than short loops. Therefore, this model is sometimes referred to as the forgetful weakly self-avoiding walk. Note that H_n is, for $p \neq 0$, *not* a functional of the local times in (1.3). Therefore, the previous approach cannot be used. Again, the question for the large- n asymptotics of the expected end-to-end distance of the polymer, $E_{Q_n} |S_n|$, is of interest. This model contains some flavor of the two-dimensional polymer model since it requires some control on the structure of the set of time-pairs at which self-intersections occur (not only on their number).

Caracciolo *et al.* [CPP94] conjecture that the critical exponent $\nu(1) = \nu(1, p)$ in (1.24) assumes the values

$$\nu(1) = \begin{cases} \min\{2 - p, 1\} & \text{for } p \leq \frac{3}{2} \\ \frac{1}{2} & \text{for } p \geq \frac{3}{2} \end{cases} \quad (2.8)$$

Hence, one expects that the polymer behaves diffusively for $p > \frac{3}{2}$ (with logarithmic corrections for $p = \frac{3}{2}$) and ballistically for $p \leq 1$, while the critical exponent interpolates linearly in between. There are two partial proofs for these conjectures: for $p > \frac{3}{2}$ and small β , and for $p \leq 1$ and large β . (The intermediate regime $p \in (1, \frac{3}{2})$ is considered much more difficult.) In both cases, the so-called *lace expansion* proved to be useful. The lace expansion is a diagrammatic resummation technique that produces a recursive relation for the Fourier transform of certain relevant quantities. See [MS93] Chapters 5 and 6 for a more detailed exposition of the lace expansion and an application to the (weakly) self-avoiding walk in dimension $d \geq 5$. The method recently received new attention when the idea was incorporated not to use this relation in order to analyze the Laplace transforms, but instead drive a carefully formulated induction hypothesis in the parameter n (see [HHS98]). Nevertheless, an essential assumption for the method to work is still a very simple path measure to which one compares the model under interest, which enforces a sufficiently small choice of the interaction parameter β .

In the diffusive case, van der Hofstad, den Hollander and Slade [HHS98] prove the following.

Theorem 2.4. For all $p > \frac{3}{2}$, there exists a $\beta_0 = \beta_0(p)$ such that for all $\beta \in (0, \beta_0(p)]$

$$\lim_{n \rightarrow \infty} Q_n^{\beta, p} \left(\frac{S_n}{\sigma \sqrt{n}} \leq C \right) = \mathcal{N}((-\infty, C]) \quad \text{for every } C \in \bar{\mathbb{R}} \quad (2.9)$$

The proof uses the lace expansion in the traditional setting which compares the forgetful weakly self-avoiding walk to simple random walk, which is a simple path measure. For this, we need the interaction to be local, which technically turns out to mean that $\sum_{n \in \mathbb{N}} n^{1-p} c_n(0) e^{r^* n} < \infty$. Here $c_n(x)$ is the expected contribution to the normalizing constant of n -step paths ending precisely in $x \in \mathbb{Z}$, and r^* is the exponential rate of the normalizing constant. Thus, if we have diffusive behaviour, then we expect that $c_n(0) e^{r^* n} \sim n^{-1/2}$, and the summation criterion becomes $\sum_n n^{(1-2p)/2} < \infty$, or $p > 3/2$. The above criterion turns out to be sharp. Indeed, the lace expansion gives that for the Fourier transform $\hat{c}_{n+1}(k)$ of $c_{n+1}(x)$ we have the following recursion equation

$$\hat{c}_{n+1}(k) = \cos(k) \hat{c}_n(k) + \sum_{m=2}^n \hat{\pi}_m(k) \hat{c}_{n+1-m}(k), \quad k \in [-\pi, \pi] \quad (2.10)$$

where $\pi_m(x)$ is a correction term whose smallness is important for the method. Note that $\cos(k)$ is the Fourier transform of the distribution of a single step of simple random walk, so that the first term corresponds to ignoring the interaction of the first step with the other n steps. The term $\pi_m(x)$ corrects for the intersections between the first step and the other n steps. The main work in [HHS98] consists of showing that, for $p > 3/2$ and small $\beta > 0$, we have that $\hat{\pi}_m(k)$ is small with respect to $\hat{c}_m(k)$. Hence, we can think of the above model as a small perturbation of simple random walk, and (2.10) allows for an induction in n .¹² For $p \in [0, 1)$, on the other hand, Kennedy [Ke94] proved a weaker version of Theorem 1.1 using a renormalization type argument. Indeed, he showed that for any $\varepsilon \in (0, 1)$ there exists a $\beta_0 = \beta_0(\varepsilon) \in (0, \infty)$ such that for all $\beta \geq \beta_0$ the probability of $\{|S_n| > (1 - \varepsilon)n\}$ under $Q_n^{\beta, p}$ converges to one as $n \rightarrow \infty$. We expect that the

¹² In fact, Theorem 2.4 holds in general dimension (as long as $p > \frac{d-4}{2}$) as the summation criterion suggests. This result in particular shows that the weakly self-avoiding walk behaves diffusively in $d \geq 5$. Hara and Slade [HS92a] in fact show diffusive behavior for *strictly* self-avoiding walk, and this proof is quite delicate and computer assisted.

method used by Kennedy can be extended to show that for any $\beta > 0$ there exists an $\varepsilon > 0$ such that the probability of $\{|S_n| > \varepsilon n\}$ under $Q_n^{\beta, p}$ converges to one.

Van der Hofstad [vdH00] extends the previous result to a full central limit theorem, and includes the critical value $p = 1$:

Theorem 2.5. For all $p \in [0, 1]$, there exists a $\beta_0 = \beta_0(p) \in (0, \infty)$ such that for all $\beta \geq \beta_0$, the central limit theorem in (1.5) holds with some appropriate $\theta^* = \theta^*(\beta, p) \in (0, 1)$ and $\sigma^* = \sigma^*(\beta, p) \in (0, \infty)$.

Van der Hofstad uses an adaptation of the lace expansion described above to compare the forgetful weakly self-avoiding walk model for large β to the strictly self-avoiding walk ($\beta = \infty$), which is an extremely simple process in dimension one. Here $e^{-\beta}$ is the parameter that has to be chosen small enough in order to make the method work.

It is difficult to adapt the Markovian method described in Section 1.3 to the forgetful weakly self-avoiding walk since one needs information about the time that has elapsed between self-intersections, which is much more than the local times in (1.3), and one seems to be unable to do that. A second serious difficulty arises from the fact that the path properties seem to be determined on scale $e^{\theta(n^{1-p})}$ rather than $e^{\theta(n)}$, as the proof of Theorem 2.5 in [vdH00] suggests. This is the content of the next conjecture:

Conjecture 2.6. For every $p \in (0, 1]$ and $\beta > 0$, the rate function I in (1.7) exists on $[-1, 1]$. Furthermore, there exists a $\theta^* = \theta^*(\beta, p) \in (0, 1)$ with the following properties.

- (i) $\lim_{n \rightarrow \infty} \frac{1}{n} E_n^{\beta, p}(|S_n|) = \theta^*$,
- (ii) I is identical to zero on $[-\theta^*, \theta^*]$ and strictly positive outside this interval,
- (iii) For any $\theta \in [-\theta^*, \theta^*]$,

$$-\lim_{n \rightarrow \infty} \frac{1}{n^{1-p}} \log Q_n^{\beta, p}(S_n \approx \theta n) = \frac{\beta}{2(1-p)\theta^{*2-p}} (\theta^* - |\theta|)^{1-p} \quad (2.11)$$

A heuristic explanation for Conjecture 2.6 (iii) is the following. For $0 \leq \theta < \theta^*$, the best strategy for the path is to first move up with speed θ^* up to time $\frac{1+\theta}{2\theta^*} n$, and then to move down with speed $-\theta^*$ up to time n , so as to arrive at θn . Paths with speed θ^* visit every point roughly $\frac{1}{\theta^*}$ times. Hence, the interaction between the two parts of the path is roughly equal to n^{1-p} times the r.h.s. of (2.11). However, this term is equal to the extra cost in comparison to a path that has precisely slope θ^* .

3. RELATED MODELS

3.1. Branching Polymer

The weakly self-avoiding walk serves as a model for a polymer that is linear in structure, since the building blocks of the polymer can make at most two connections to other building blocks. We next define a model that allows for more connections, and serves as a model for *branching polymers*.

We first define the Poisson branching process conditioned on a fixed family size n . We start with a single individual having ξ offspring, where ξ is a Poisson random variable of mean 1, i.e., $\mathbb{P}(\xi = m) = (em!)^{-1}$. Each of the offspring then independently has offspring of its own, with the same critical Poisson distribution. This defines the family tree T . We are interested in trees of size $|T| = n$. This will correspond to polymers of a fixed number of building blocks. These trees are completely determined by the set of offspring numbers $(\xi_i)_{i \in T}$, where ξ_i is the number of offspring of particle $i \in T$.

We next define the branching random walk. This is a random map S mapping each individual of the family tree T to \mathbb{Z}^d , such that the root is mapped to the origin and adjacent vertices in the tree are mapped to nearest neighbors in \mathbb{Z}^d . Given the family tree T , each random map S is equally likely. The reference measure on configurations (T, S) is then given by¹³

$$P_n(T, S) = \frac{1}{(2d)^{n-1}} \frac{n^{n-1}}{n!} \prod_{i \in T} \frac{1}{\xi_i!} \quad (3.1)$$

We will refer to the pair (T, S) as an *embedded tree in \mathbb{Z}^d* .

The branching polymer model is now formally defined as in (1.1), where the Hamiltonian $H_n = H_n(T, S)$ is given by

$$H_n(T, S) = \beta \sum_{i, j \in T: i \neq j} \mathbb{1}\{S_i = S_j\} \quad (3.2)$$

The measure Q_n^β on the set of embedded n -site trees rewards self-avoidance by giving a penalty $e^{-\beta}$ to each self-intersection of an embedded tree, i.e., to each pair $i \neq j \in T$ with $S_i = S_j$.

¹³ To see that this is indeed Poisson branching random walk measure conditioned on tree size n , note that for a Poisson(1) branching process we have that $P(T) = \prod_{i \in T} \frac{e^{-1}}{\xi_i!}$, and $P(|T| = n) = \frac{n^{n-1}}{n!} e^{-n}$ (see e.g., [BCHS99]).

In [BCHS99] it is shown that, for fixed n and as $\beta \rightarrow \infty$, the measure Q_n^β converges towards a distribution Q_n^∞ on embedded trees in \mathbb{Z}^d having no self-intersections, i.e., on pairs (T, S) such that $S_i \neq S_j$ for all distinct $i, j \in T$. Moreover, this measure turns out to be uniform in the following sense. A *lattice tree* is a finite connected set of bonds that contains no cycles. Here, a *bond* in \mathbb{Z}^d is a pair $\{x, y\}$ of sites $x, y \in \mathbb{Z}^d$ with $\|x - y\|_1 = 1$. If we sum out $Q_n(T, S)$ over all combinations (T, S) such that the image of T under S is a fixed lattice tree L of size n in \mathbb{Z}^d , then the resulting measure on lattice trees is the uniform measure.¹⁴ The above construction is in the spirit of the self-avoiding walk. If we take the limit of $\beta \rightarrow \infty$ for the weakly self-avoiding walk, then the resulting measure is the uniform measure on all n -step self-avoiding paths. Hence, we can think of lattice trees as branching self-avoiding walks. Therefore, we will refer to the above weakly interacting model as the *branching weakly self-avoiding walk*.

Define R_n to be the number of distinct points visited by the branching weakly self-avoiding walk. It is easy to adapt the argument for the weakly self-avoiding walk to show that with overwhelming probability $R_n \geq \epsilon n$ for some $\epsilon > 0$ sufficiently small. We have the following more precise conjecture:

Conjecture 3.1. Fix $d=1$. For all $\beta \in (0, \infty)$, there exists $\theta^* = \theta^*(\beta) \in (0, 1)$, such that, for every $\epsilon > 0$,

$$Q_n^\beta \left(\left| \frac{R_n}{n} - \theta^* \right| > \epsilon \right) = 0 \quad (3.3)$$

Moreover, $\lim_{\beta \downarrow 0} \theta^*(\beta) \beta^{-3/7}$ exists in $(0, \infty)$.

Let us give a heuristic explanation of the scaling described above. As in Section 1.2, $\frac{\beta}{\theta} n$ is the minimal value of the Hamiltonian in (3.2) when the range is θn . The probability that the range is of order θn is, for small θ , approximately equal to $e^{-\theta^{4/3} n}$.¹⁵ The optimal θ we find as in Section 1.2 by maximizing the exponential rate of $e^{-n[\beta/\theta + \theta^{4/3}]}$, i.e., the contribution from the Hamiltonian given that the range is θn times the probability to have range θn . This gives that $\theta^*(\beta) \sim \beta^{3/7}$, and that the exponential rate of the normalizing constant scales as $r^*(\beta) \sim \beta^{4/7}$.

¹⁴ This is true only for a Poisson offspring distribution.

¹⁵ This is consistent with the fact that the *typical* size of the range is $n^{1/4}$ (see e.g., [BCHS99]), since substituting $\theta = n^{-3/4}$ gives that this probability is $\mathcal{O}(1)$. Moreover, the above asymptotics can be proved for critical binary branching by using the analogy of binary branching to one-dimensional random walk excursions. Any finite variance critical branching is expected to lie in the same universality class.

There has been considerable progress also in high dimensions, where it has been proved that the limit is integrated super-Brownian excursion (ISE), which is a random probability measure and can be regarded as super-Brownian motion conditioned on total mass equal to one (see [HS94, DS98, DS97] and the references therein). ISE is known to arise as the scaling limit for $\beta=0$ when space is scaled by a factor $n^{1/4}$ as $n \rightarrow \infty$ in all dimensions (see e.g., [BCHS99]). In this respect, ISE plays a similar role for branching random walk conditioned on fixed family size as Brownian motion does for simple random walk.

There should be a close connection between the above scaling conjecture for small β to a continuous-time model on the basis of ISE as for the weakly self-avoiding walk (see Section 1.4). Indeed, consider super-Brownian motion conditioned on total mass equal to $T > 0$ which is a scaled version of ISE. This process has a density in \mathbb{R} . Now we define a Hamiltonian which is β times the integral over the square of the density of this process. The resulting measure is a continuous space/time polymer model like the branching weakly self-avoiding walk. We call this model the *branching Edwards model*. We conjecture that, as T tends to infinity, the support of the random measure in this model satisfies a law of large numbers with speed equal to $b\beta^{3/7}$, and the scaling of the exponential rate of the normalizing constant is $a\beta^{4/7}$. The constants a and b should be the scaling limits of $\theta(\beta)$ and $r(\beta)$ in the branching weakly self-avoiding walk.

3.2. Charged Polymer

Let $\omega = (\omega_i)_{i \in \mathbb{N}_0}$ be an i.i.d. sequence of centered random variables with variance one, which play the role of an electrical charges of the molecules. We use the (random) Hamiltonian

$$H_n^\omega = \beta \sum_{\substack{i, j=0 \\ i \neq j}}^n \omega_i \omega_j \mathbb{1}\{S_i = S_j\} \quad (3.4)$$

which is the total electrical interaction charge of the polymer $S = (S_0, \dots, S_n)$: opposite charges of pairs of two molecules at the same site give a negative contribution, whereas equal charges give a positive contribution. We again define the polymer measure $Q_n = Q_n^\omega$ (which is now random) by (1.1).¹⁶ Again we ask for the large- n behavior of $|S_n|$ under Q_n^ω , this time almost surely with respect to the charge configuration ω . This is the so-called *quenched* setting, in contrast to the *annealed* one, where one

¹⁶ A continuous variant of this model is considered by Buffet and Pule (1997) in [BP97].

averages out over the charges. (Physicists may consider the quenched setting more realistic.)

To the best of our knowledge nothing is known about the quenched problem, and we also cannot offer any conjecture. On the other hand, in the annealed setting there is a result that occurred as the by-product in a study of the parabolic Anderson model by Biskup and König in [BK00]. Denote expectation with respect to ω by $\langle \cdot \rangle$. Note that the mean of $e^{-H_n^\omega}$ with respect to ω can be expressed in terms of the local times in (1.3) as

$$\langle e^{-H_n^\omega} \rangle = \exp \left\{ \sum_{x \in \mathbb{Z}^d} F(\ell_n(x)) \right\}, \quad \text{where } F(l) = \log \left\langle \exp \left\{ -\beta \left[\sum_{i=1}^l \omega_i \right]^2 \right\} \right\rangle$$

According to the invariance principle, $\sum_{i=1}^l \omega_i$ is approximately normal with mean 0 and variance l . This implies that $F(l) = \frac{1}{2} \log(1 + \beta l)(1 + o(1)) = \frac{1}{2} \log l(1 + o(1))$ as $l \rightarrow \infty$. Hence, the annealed partition sum $E_n \langle e^{-H_n^\omega} \rangle$ falls into the class of models analyzed in [BK00], and the result reads:

$$E_n \langle e^{-H_n^\omega} \rangle = \exp \{ -r^* n^{d/(d+2)} (\log n)^{2/(d+2)} (1 + o(1)) \}, \quad n \rightarrow \infty \quad (3.5)$$

Here $r^* \in (0, \infty)$ is independent of β .

The proof suggests that under the annealed measure the end-to-end distance $|S_n|$ behaves like $(\frac{n}{\log n})^{1/(d+2)}$. We can see this heuristically in the following way. Assume that $S_n \sim \alpha_n$ for some deterministic sequence $\alpha_n \rightarrow \infty$. Assuming homogeneity, the local times are of order n/α_n^d , so that $\sum_x F(\ell_n(x)) \approx \alpha_n^d \log(n/\alpha_n^d)$. We easily see that the interaction is self-attractive, so that we assume that $\alpha_n = o(\sqrt{n})$. Then the probability that $S_n \sim \alpha_n$ is roughly $e^{-n/2\alpha_n^2}$. Hence, we find that the appropriate α_n is obtained by maximizing $e^{-[\alpha_n^d \log(n\alpha_n^{-d}) + n/2\alpha_n^2]}$, yielding $\alpha_n = (\frac{n}{\log n})^{1/(d+2)}$ (const. + $o(1)$) and suggesting (3.5).

3.3. Heteropolymer

Let $\omega = (\omega_i)_{i \in \mathbb{N}}$ be an i.i.d. sequence of random variables taking values ± 1 with probability $\frac{1}{2}$. For $n \in \mathbb{N}$, define the (random) measure $Q_n = Q_n^{\lambda, h, \omega}$ on n -step paths via (1.1) by setting

$$H_n(S) = \lambda \sum_{i=1}^n (\omega_i + h) \text{sign}(S_i) \quad (3.6)$$

where $h \in [0, 1)$, $\lambda \in [0, \infty)$ are parameters.

Equation (3.6) models a polymer near an interface in the following way. Think of $(i, S_i)_{i=0}^n$ as a directed polymer consisting of $n+1$ building blocks, where time is viewed as an extra spatial dimension. Think of the upper half plane as “oil” and the lower half plane as “water.” There are two types of building blocks, occurring randomly with equal probability and indexed by ω : $\omega_i = +1$ means that building block i is attracted by “water,” while $\omega_i = -1$ means that building block i is attracted by “oil.” For $h \in (0, 1)$, the polymer has an overall tendency to prefer “oil,” so that S_i is more likely to be positive.

Bolthausen and den Hollander [BoH97] investigate this model in dimension one. They define the polymer to be *localized* if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n^{\lambda, h, \omega} > \lambda h \quad (3.7)$$

and *delocalized* when the limit equals λh . (The limit exists ω -a.s. and is non-random.) They prove that there is a phase transition, i.e., for every $\lambda > 0$ there exists a critical value $h_c = h_c(\lambda)$ such that there is localization for $0 \leq h < h_c$, while there is delocalization for $h \geq h_c$. Moreover, they prove some bounds and asymptotics of $\lambda \mapsto h_c(\lambda)$. Intuitively, in the localized regime the polymer wobbles around the interface with high probability. In the delocalized regime, on the other hand, the frequency of time the polymer spends in the “oil” converges to 1 (see Biskup and den Hollander [BiH97]). Note that if the polymer is always in the upper half plane, then we have equality in (3.7), since in that case $\text{sign}(S_i) = 1$ for all i and $\sum_{i=1}^n \omega_i = o(n)$ a.s. by the law of large numbers applied to ω . Furthermore, Biskup and den Hollander [BiH97] prove that in the localized regime the tails of the distribution of the height of the path at any time in between 0 and n are exponentially small. Furthermore, the interface is visited a positive fraction of the time. This result was already proved for $h=0$ by Sinai [Si93].

In the delocalized regime, where the path spends most of its time above the interface, it is expected that the law of $(n^{-1/2} S_{[tn]})_{t \in [0, 1]}$ under \mathcal{Q}_n converges weakly. Indeed, since the law of (S_0, \dots, S_n) under \mathcal{Q}_n only depends upon the signs of S_0, \dots, S_n , this property should follow from the well-known conditional weak convergence of $(n^{-1/2} S_{[tn]})_{t \in [0, 1]}$ under the free random walk given that $S_i > 0$ for all $0 < i \leq n$.

3.4. Myopic Self-Avoiding Walk

A mechanism for a polymer measure different from the model in (1.1), the so-called *myopic random walk*, is defined in one dimension as follows. Given a parameter $\beta \in (0, \infty)$, conditionally on the path (S_0, \dots, S_n) , we

define S_{n+1} as $S_n - 1$ or $S_n + 1$ with probability proportional to $e^{-\beta \ell_n(S_n-1)}$ and $e^{-\beta \ell_n(S_n+1)}$, respectively. In words, the polymer chain evolves step by step randomly and chooses one of the two nearest neighbors of the current position with probability proportional to the negative exponential of the current number of molecules in the two neighboring sites. Let Q_n be the distribution of the chain (S_0, \dots, S_n) . Then $(Q_n)_{n \in \mathbb{N}}$ is a *consistent* family of path measures (i.e., Q_n is the projection of Q_{n+1} on n -step paths) and is a model for a growing polymer chain.¹⁷ The myopic self-avoiding walk models the situation in which the polymer growth is much more rapid than the thermal motion of the building blocks of the polymer, so that the polymer does not have the opportunity to rearrange itself during its growth.

Note that the strictly self-avoiding version of this model (i.e., the model with $\beta = \infty$) is not well-defined in general dimension, since the chain may “get stuck in a trap” where it is not possible to avoid self-intersections in future. The appropriate version for $\beta = \infty$ should be the model that arises by taking $\beta \rightarrow \infty$. In this model (which is not self-avoiding) the respective steps of the chain are chosen uniformly from the set of neighbors with minimal local time.

See [PP87] for non-rigorous scaling arguments and numerical simulations, which suggest that $E_{Q_n} |S_n| \approx n^{\nu(d)}$ for large n , where $\nu(1) = \frac{2}{3}$ and $\nu(d) = \frac{1}{2}$ for $d \geq 2$. Hence, the self-repellence in the myopic model is much weaker than for weakly self-avoiding walk. This is intuitively clear, because the growing polymer does not rearrange itself, so that it gets stuck in an energetically suboptimal shape.

Tóth [To95] investigates a version of the myopic walk where the choice of the two neighbors is not made according to the *site* local times, but the *bond* local times, i.e., the number of previous jumps along the bonds between S_n and $S_n \pm 1$. In this model, it turns out that a variant of the Markovian description for the sequence of up-step numbers $m(x) = \{0 < i \leq T : S_{i-1} = x-1, S_i = x\}$ at a certain stopping time T applies, as in Section 1.3 above. This nice observation allows for a proof of a scaling limit law for the end-to-end distance at certain independent geometrical random times whose parameter goes to infinity (or, equivalently, for the Laplace transforms of the properly scaled endpoint). More precisely, Tóth proves a result which suggests that $n^{-2/3} S_n$ converges weakly to some non-trivial distribution which is identified analytically. He also conjectures that the appropriately scaled path converges weakly. Tóth and Werner [TW98] construct a candidate for the limit process, the *self-repelling motion*, which should be seen as the continuous analog of the above

¹⁷ This model is sometimes called the “true” self-avoiding walk.

model. This process has interesting path properties not shared by Brownian motion or the Edwards model. The proof that this process is really the limiting process of the discrete model has not been tackled yet.

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